

# A dipolar droplet bound in a trapped Bose-Einstein condensate

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We study the statics and dynamics of a dipolar Bose-Einstein condensate (BEC) droplet bound by inter-species contact interaction in a trapped non-dipolar BEC. Our findings are demonstrated in terms of stability plots of a dipolar  $^{164}\text{Dy}$  droplet bound in a trapped non-dipolar  $^{87}\text{Rb}$  BEC with a variable number of  $^{164}\text{Dy}$  atoms and the inter-species scattering length. A trapped non-dipolar BEC of a fixed number of atoms can only bind a dipolar droplet containing atoms less than a critical number for the inter-species scattering length between two critical values. The shape and size (statics) as well as the small breathing oscillation (dynamics) of the dipolar BEC droplet are studied using the numerical and variational solutions of a mean-field model. We also suggest an experimental procedure for achieving such a  $^{164}\text{Dy}$  droplet by relaxing the trap on the  $^{164}\text{Dy}$  BEC in a trapped binary  $^{87}\text{Rb}$ - $^{164}\text{Dy}$  mixture.

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## I. INTRODUCTION

The experimental observation of a dipolar Bose-Einstein condensate (BEC) of  $^{52}\text{Cr}$  [1–5],  $^{164}\text{Dy}$  [6, 7] and  $^{168}\text{Er}$  [8] atoms with large magnetic dipole moments has opened new directions of research in cold atoms in the quest of novel and interesting features related to the anisotropic long-range dipolar interaction. Polar molecules, with much larger (electric) dipole moment, are also being considered [9] for BEC experiments. The atomic interaction in a dilute BEC of alkali-metal and other types of atoms (with negligible dipole moment) is represented by an S-wave contact (delta-function) potential. However, the non-local anisotropic long-range dipolar interaction acts in all partial waves and is attractive in certain directions and repulsive in others.

An untrapped three-dimensional (3D) BEC with attractive interaction does not exist in nature due to collapse instability [10]. However, the collapse of an untrapped BEC can be avoided due to interspecies contact interaction in a binary mixture with a trapped BEC. The shape of such a stable droplet bound in a trapped BEC is controlled by the inter-species interactions, whereas that of a trapped BEC is determined by the underlying trap. Here we consider such a dipolar droplet bound in a trapped nondipolar BEC. The effect of the underlying atomic interaction, specially that of the anisotropic dipolar interaction, will easily manifest in such a dipolar droplet. These droplets will be termed quasi-free as they can easily move around inside the larger trapped BEC responsible for their binding. By taking the trapped BEC to be nondipolar, one can easily study the effect of intra-species dipolar interaction on the quasi-free droplet in the absence of any inter-species dipolar interaction.

Dipolar BECs are immediately distinguishable from

those with purely contact interactions by their strong shape and stability dependence [11] on trapping geometry. Here, we are proposing a new way to trap a dipolar BEC in the form of a quasi-free dipolar droplet, using an attractive inter-species mean-field potential, which introduces a unique trapping geometry that results in further interesting shape and stability characteristics of dipolar BECs. We study the statics and dynamics of a quasi-free dipolar droplet using numerical solution and variational approximation of a mean-field model [12, 13]. For this purpose we consider a binary mixture of non-dipolar  $\text{Rb}87$  and dipolar  $\text{Dy}164$  where the trapped non-dipolar  $^{87}\text{Rb}$  BEC could be in a cigar- or disk-shape. Among the available dipolar BECs [4, 7, 8],  $^{164}\text{Dy}$  atoms have the largest (magnetic) dipolar interaction strength [1, 6]. The existence of stable 3D  $^{164}\text{Dy}$  droplets is illustrated by stability plots involving the number of atoms of the two species and the inter-species scattering length. For a fixed number  $N(\text{Rb})$  of  $^{87}\text{Rb}$  atoms the dipolar droplet could be bound below a critical number  $N(\text{Dy})$  of  $^{164}\text{Dy}$  atoms between two limiting values of inter-species attraction. If the inter-species attraction is too small, the  $^{164}\text{Dy}$  atoms in the droplet cannot be bound and, for a very large inter-species attraction, the droplet is destroyed by collapse instability. Usually, the dipolar droplet has the same shape as the trap acting on the non-dipolar BEC. However, for a large number of  $^{164}\text{Dy}$  atoms, as the inter-species attraction approaches the collapse instability, the dipolar droplet always is of cigar shape even if the external trap is disk shaped and eventually the dipolar droplet collapses on the axial  $z$  axis from the cylindrical configuration. For a small number of atoms, the dipolar droplet collapses to the center maintaining its shape rather than on the axial  $z$  axis.

The variational approximation to the sizes and chemical potentials of the stationary droplets is compared with the numerical solution of the mean-field model. The numerical study of breathing oscillation of the stable dipolar droplet is found to be in reasonable agreement with a time-dependent variational model calculation. We

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also demonstrate a viable experimental way of creating a  $^{164}\text{Dy}$  droplet bound in a trapped  $^{87}\text{Rb}$  BEC, e.g., by slowly removing the trap on a trapped binary  $^{164}\text{Dy}$ - $^{87}\text{Rb}$  mixture, while the trapped  $^{164}\text{Dy}$  BEC evolves into a quasi-free droplet.

In Sec. II the mean-field model for a dipolar BEC droplet bound in a trapped non-dipolar BEC is developed. A time-dependent, analytic, Euler-Lagrange Gaussian variational approximation of the model is also presented. The results of numerical calculation are shown in Sec. III. Finally, in Sec. IV we present a brief summary of our findings.

## II. MEAN-FIELD MODEL FOR A QUASI-FREE DIPOLAR DROPLET

We consider a binary BEC, where one of the species is dipolar and the other non-dipolar, interacting via inter- and intra-species interactions with the mass, number, magnetic /colored dipole moment, and scattering length for the two species  $i = 1, 2$ , denoted by  $m_i, N_i, \mu_i, a_i$ , respectively. The first species ( $^{87}\text{Rb}$ ) is taken to be non-dipolar ( $\mu_1 = 0$ ) and trapped while the second species ( $^{164}\text{Dy}$ ) is dipolar ( $\mu_2 \neq 0$ ) and polarized along axial  $z$  direction. The angular frequencies for the axially-symmetric trap on  $^{87}\text{Rb}$  along  $x, y$  and  $z$  directions are taken as  $\omega_x = \omega_y = \omega_1$  and  $\omega_z = \lambda_1 \omega_1$ . The inter- ( $V_{12}$ ) and intra-species ( $V_i$ ) interactions for two atoms at positions  $\mathbf{r}$  and  $\mathbf{r}'$  are taken as

$$V_{12}(\mathbf{R}) = \frac{2\pi\hbar^2 a_{12}}{m_R} \delta(\mathbf{R}), \quad V_1(\mathbf{R}) = \frac{4\pi\hbar^2 a_1}{m_1} \delta(\mathbf{R}), \quad (1)$$

$$V_2(\mathbf{R}) = \frac{\mu_0 \mu_2^2}{4\pi} \frac{1 - 3 \cos^2 \theta}{|\mathbf{R}|^3} + \frac{4\pi\hbar^2 a_2}{m_2} \delta(\mathbf{R}), \quad (2)$$

where  $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ ,  $\mu_0$  is the permeability of free space,  $\theta$  is the angle made by the vector  $\mathbf{R}$  with the polarization  $z$  direction, and  $m_R = m_1 m_2 / (m_1 + m_2)$  is the reduced mass of the two species of atoms. With these interactions, the coupled Gross-Pitaevskii (GP) equations for the binary dipolar BEC can be written as [13]

$$\begin{aligned} i\hbar \frac{\partial \phi_1(\mathbf{r}, t)}{\partial t} = & \left[ -\frac{\hbar^2}{2m_1} \nabla^2 + \frac{1}{2} m_1 \omega_1^2 (\rho^2 + \lambda_1^2 z^2) \right. \\ & \left. + \frac{4\pi\hbar^2}{m_1} a_1 N_1 |\phi_1(\mathbf{r}, t)|^2 + \frac{2\pi\hbar^2}{m_R} a_{12} N_2 |\phi_2(\mathbf{r}, t)|^2 \right] \phi_1(\mathbf{r}, t), \end{aligned} \quad (3)$$

$$\begin{aligned} i\hbar \frac{\partial \phi_2(\mathbf{r}, t)}{\partial t} = & \left[ -\frac{\hbar^2}{2m_2} \nabla^2 \right. \\ & + \frac{4\pi\hbar^2}{m_2} a_2 N_2 |\phi_2(\mathbf{r}, t)|^2 + \frac{2\pi\hbar^2}{m_R} a_{12} N_1 |\phi_1(\mathbf{r}, t)|^2 \\ & \left. + N_2 \frac{\mu_0 \mu_2^2}{4\pi} \int V_{dd}(\mathbf{R}) |\phi_2(\mathbf{r}', t)|^2 d\mathbf{r}' \right] \phi_2(\mathbf{r}, t), \end{aligned} \quad (4)$$

$$V_{dd}(\mathbf{R}) = \frac{1 - 3 \cos^2 \theta}{R^3}, \quad \rho^2 = x^2 + y^2, \quad i = \sqrt{-1}. \quad (5)$$

To compare the dipolar and contact interactions, the intra-species dipolar interaction is expressed in terms of the length scale  $a_{dd}$ , defined by  $a_{dd} = \mu_0 \mu_2^2 m_2 / (12\pi\hbar^2)$ . We express the strength of the dipolar interaction in Eq. (4) by this length scale and transform Eqs. (3) and (4) into the following dimensionless form [13]

$$i \frac{\partial \phi_1(\mathbf{r}, t)}{\partial t} = \left[ -\frac{\nabla^2}{2} + \frac{1}{2} (\rho^2 + \lambda_1^2 z^2) + g_1 |\phi_1|^2 \right. \\ \left. + g_{12} |\phi_2|^2 \right] \phi_1(\mathbf{r}, t), \quad (6)$$

$$i \frac{\partial \phi_2(\mathbf{r}, t)}{\partial t} = \left[ -m_{12} \frac{\nabla^2}{2} + g_2 |\phi_2|^2 + g_{21} |\phi_1|^2 \right. \\ \left. + g_{dd} \int V_{dd}(\mathbf{R}) |\phi_2(\mathbf{r}', t)|^2 d\mathbf{r}' \right] \phi_2(\mathbf{r}, t), \quad (7)$$

where  $m_{12} = m_1/m_2$ ,  $g_1 = 4\pi a_1 N_1$ ,  $g_2 = 4\pi a_2 N_2 m_{12}$ ,  $g_{12} = 2\pi m_1 a_{12} N_2/m_R$ ,  $g_{21} = 2\pi m_1 a_{12} N_1/m_R$ ,  $g_{dd} = 3N_2 a_{dd} m_{12}$ . In Eqs. (6) and (7), length is expressed in units of oscillator length of the first species  $l_0 = \sqrt{\hbar/m_1 \omega_1}$ , energy in units of oscillator energy  $\hbar \omega_1$ , density  $|\phi_i|^2$  in units of  $l_0^{-3}$ , and time in units of  $t_0 = \omega_1^{-1}$ .

Convenient analytic variational approximation to Eqs. (6) and (7) can be obtained with the following ansatz for the wave functions [14–16]

$$\phi_i(\mathbf{r}, t) = \frac{\pi^{-3/4}}{w_{\rho i} \sqrt{w_{z i}}} \exp \left[ -\frac{\rho^2}{2w_{\rho i}^2} - \frac{z^2}{2w_{z i}^2} + i\alpha_i \rho^2 + i\beta_i z^2 \right] \quad (8)$$

where  $\mathbf{r} = \{\vec{\rho}, z\}$ ,  $\vec{\rho} = \{x, y\}$ ,  $w_{\rho i}$  and  $w_{z i}$  are the widths and  $\alpha_i$  and  $\beta_i$  are additional variational parameters. The effective Lagrangian for the binary system is

$$\begin{aligned} L = & \int d\mathbf{r} \frac{1}{2} \left[ \sum_i \left\{ iN_i (\phi_i \dot{\phi}_i^* - \phi_i^* \dot{\phi}_i) + N_i g_i |\phi_i(\mathbf{r})|^4 \right\} \right. \\ & + N_1 |\nabla \phi_1(\mathbf{r})|^2 + m_{12} N_2 |\nabla \phi_2(\mathbf{r})|^2 + N_1 (\rho^2 + \lambda_1^2 z^2) \\ & \times |\phi_1(\mathbf{r})|^2 \left. \right] + N_1 g_{12} \int d\mathbf{r} |\phi_1(\mathbf{r})|^2 |\phi_2(\mathbf{r})|^2 \\ & + \frac{N_2}{2} g_{dd} \int \int V_{dd}(\mathbf{R}) |\phi_2(\mathbf{r}')|^2 |\phi_2(\mathbf{r})|^2 d\mathbf{r}' d\mathbf{r}, \quad (9) \\ = & \sum_{i=1}^2 \frac{N_i}{2} (2w_{\rho i}^2 \dot{\alpha}_i + w_{z i}^2 \dot{\beta}_i) + \frac{N_1}{2} \left[ \frac{1}{w_{\rho 1}^2} + \frac{1}{2w_{z 1}^2} \right. \\ & + 4w_{\rho 1}^2 \alpha_1^2 + 2w_{z 1}^2 \beta_1^2 \left. \right] + \frac{N_2 m_{12}}{2} \left[ \frac{1}{w_{\rho 2}^2} + \frac{1}{2w_{z 2}^2} \right. \\ & + 4w_{\rho 2}^2 \alpha_2^2 + 2w_{z 2}^2 \beta_2^2 \left. \right] + \frac{N_1}{2} \left[ w_{\rho 1}^2 + \lambda_1^2 \frac{w_{z 1}^2}{2} \right] \\ & + \frac{N_1^2 a_1}{\sqrt{2\pi} w_{\rho 1} w_{z 1}} + \frac{N_2^2 m_{12}}{\sqrt{2\pi} w_{\rho 2} w_{z 2}} [a_2 - a_{dd} f(\kappa)] \\ & + \frac{CN_1 N_2}{2AB^{1/2}}, \quad \kappa = \frac{w_{\rho 2}}{w_{z 2}}, \end{aligned} \quad (10)$$

$$f(\kappa) = \frac{1 + 2\kappa^2 - 3\kappa^2 d(\kappa)}{1 - \kappa^2}, \quad d(\kappa) = \frac{\text{atan}(\sqrt{\kappa^2 - 1})}{\sqrt{\kappa^2 - 1}},$$

where  $A = w_{\rho 1}^2 + w_{\rho 2}^2$ ,  $B = w_{z 1}^2 + w_{z 2}^2$  and  $C = 4a_{12}m_1/(\sqrt{\pi}m_R)$ . In these equations the overhead dot denotes time derivative. The Euler-Lagrange variational equations for the widths for the effective Lagrangian (10), obtained in usual fashion [16], are:

$$\ddot{w}_{\rho 1} + w_{\rho 1} = \frac{1}{w_{\rho 1}^3} + \frac{1}{\sqrt{2\pi}} \frac{2N_1 a_1}{w_{\rho 1}^3 w_{z 1}} + \frac{CN_2 w_{\rho 1}}{A^2 B^{1/2}}, \quad (11)$$

$$\ddot{w}_{z 1} + \lambda_1^2 w_{z 1} = \frac{1}{w_{z 1}^3} + \frac{1}{\sqrt{2\pi}} \frac{2N_1 a_1}{w_{\rho 1}^3 w_{z 1}^2} + \frac{CN_2 w_{z 1}}{AB^{3/2}}, \quad (12)$$

$$\ddot{w}_{\rho 2} = \frac{m_{12}}{w_{\rho 2}^3} + \frac{N_2 m_{12} [2a_2 - a_{dd} g(\kappa)]}{\sqrt{2\pi} w_{\rho 2}^3 w_{z 2}} + \frac{CN_1 w_{\rho 2}}{A^2 B^{1/2}}, \quad (13)$$

$$\ddot{w}_{z 2} = \frac{m_{12}}{w_{z 2}^3} + \frac{2N_2 m_{12} [a_2 - a_{dd} h(\kappa)]}{\sqrt{2\pi} w_{\rho 2}^3 w_{z 2}^2} + \frac{CN_1 w_{z 2}}{AB^{3/2}}, \quad (14)$$

$$g(\kappa) = \frac{2 - 7\kappa^2 - 4\kappa^4 + 9\kappa^4 d(\kappa)}{(1 - \kappa^2)^2}, \quad (15)$$

$$h(\kappa) = \frac{1 + 10\kappa^2 - 2\kappa^4 - 9\kappa^4 d(\kappa)}{(1 - \kappa^2)^2}. \quad (16)$$

The solution of the time-dependent Eqs. (11) – (14) gives the dynamics of the variational approximation.

If  $\mu_i$  is the chemical potential with which the stationary wave function  $\phi_i(\mathbf{r}, t)$  propagates in time, e.g.  $\phi_i(\mathbf{r}, t) \sim \exp(-i\mu_i t)\phi_i(\mathbf{r})$ , then the variational estimate for  $\mu_i$  is:

$$\begin{aligned} \mu_1 = & \frac{1}{2} \left[ \frac{1}{w_{\rho 1}^2} + \frac{1}{2w_{z 1}^2} \right] + \frac{1}{2} \left[ w_{\rho 1}^2 + \lambda_1^2 \frac{w_{z 1}^2}{2} \right] \\ & + \frac{2N_1 a_1}{\sqrt{2\pi} w_{\rho 1}^2 w_{z 1}} + \frac{CN_2}{2AB^{1/2}}, \end{aligned} \quad (17)$$

$$\begin{aligned} \mu_2 = & \frac{m_{12}}{2} \left[ \frac{1}{w_{\rho 2}^2} + \frac{1}{2w_{z 2}^2} \right] \\ & + \frac{2N_2 m_{12}}{\sqrt{2\pi} w_{\rho 2}^2 w_{z 2}} [a_2 - a_{dd} f(\kappa)] + \frac{CN_1}{2AB^{1/2}}. \end{aligned} \quad (18)$$

The energy of the system is given by

$$\begin{aligned} E = & \frac{N_1}{2} \left[ \frac{1}{w_{\rho 1}^2} + \frac{1}{2w_{z 1}^2} \right] + \frac{N_1}{2} \left[ w_{\rho 1}^2 + \lambda_1^2 \frac{w_{z 1}^2}{2} \right] \\ & + \frac{N_2 m_{12}}{2} \left[ \frac{1}{w_{\rho 2}^2} + \frac{1}{2w_{z 2}^2} \right] + \frac{N_1^2 a_1}{\sqrt{2\pi} w_{\rho 1}^2 w_{z 1}} \\ & + \frac{N_2^2 m_{12}}{\sqrt{2\pi} w_{\rho 2}^2 w_{z 2}} [a_2 - a_{dd} f(\kappa)] + \frac{CN_1 N_2}{2AB^{1/2}}. \end{aligned} \quad (19)$$

The widths of the stationary state can be obtained from the solution of Eqs. (11) – (14) setting the time derivatives of the widths equal to zero. This procedure is equivalent to a minimization of the energy (19), provided the stationary state is stable and corresponds to a energy minimum.

### III. NUMERICAL RESULTS

We solve Eqs. (6) and (7) by split-step Crank-Nicolson discretization scheme using a space step of 0.1 and the

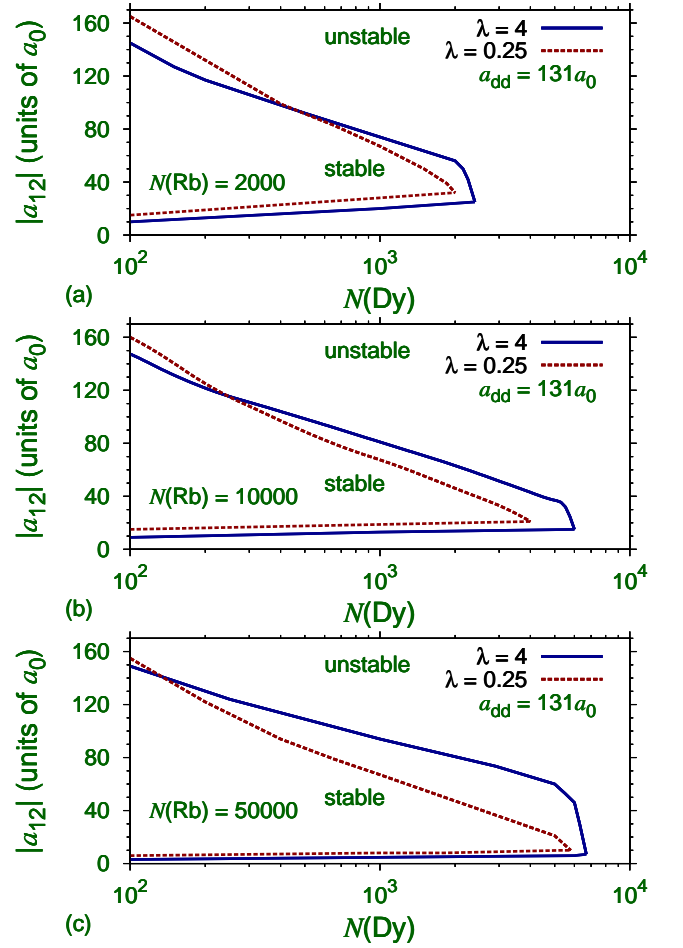


FIG. 1: (Color online)  $|a_{12}| - N(\text{Dy})$  stability plot showing the domain of stable  $^{164}\text{Dy}$  droplet in a binary  $^{87}\text{Rb}-^{164}\text{Dy}$  mixture for (a)  $N(\text{Rb}) = 2000$ , (b)  $N(\text{Rb}) = 10000$ , and (c)  $N(\text{Rb}) = 50000$ , and for  $\lambda = 0.25$  and  $4$ . The intra-species scattering lengths are taken as  $a_i = 110a_0$ .

time step 0.001 [15, 17]. The contribution of the dipolar interaction is calculated in momentum space by Fourier transformation [15]. For both species of atoms  $^{87}\text{Rb}$  and  $^{164}\text{Dy}$  we take the intra-species scattering length as  $a_i = 110a_0$ , and the strength of dipolar interaction as  $a_{dd} = 131a_0$  [7, 10]. The yet unknown inter-species scattering length  $a_{12}$  is taken as a variable. The variation of  $a_{12}$  can be achieved experimentally by the Feshbach resonance technique [18]. We consider the trap frequency  $\omega_1 = 2\pi \times 115$  Hz, so that the length scale  $l_0 \equiv \sqrt{\hbar/m_1\omega_1} = 1$   $\mu\text{m}$  and time scale  $t_0 \equiv \omega_1^{-1} = 1.38$  ms.

We find that a quasi-free  $^{164}\text{Dy}$  droplet is achievable for a moderately attractive inter-species attraction (negative  $a_{12}$ ) and for appropriate values of the number of atoms of the two species  $N_1$  and  $N_2$ . We illustrate the domain of existence of a stable  $^{164}\text{Dy}$  droplet in terms of stability plots in Figs. 1 (a), (b) and (c) for  $N(\text{Rb}) = 2000, 10000$  and  $50000$ , and for  $\lambda = 0.25$  and  $4$ . We consider cigar- and disk-shaped

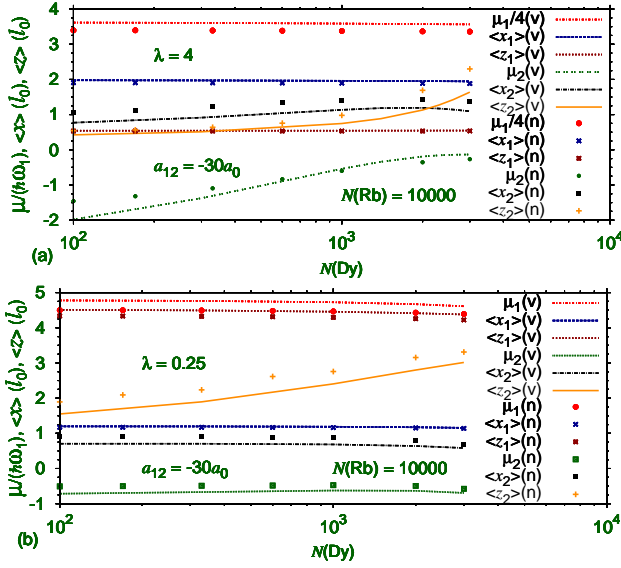


FIG. 2: (Color online) Variational (v) and numerical (n) results for chemical potential  $\mu$  and rms sizes  $\langle x \rangle, \langle z \rangle$  for the trapped  $^{87}\text{Rb}$  BEC of 10000 atoms and the bound  $^{164}\text{Dy}$  droplet versus the number of  $^{164}\text{Dy}$  atoms for (a)  $\lambda = 4$  and (b)  $\lambda = 0.25$ . The inter-species scattering length is  $a_{12} = -30a_0$ .

$^{164}\text{Dy}$  droplets in this study, where the effect of dipolar interaction is expected to be more prominent, and not spherically symmetric ones where the effect of dipolar interaction is expected to be a minimum [1, 5]. In all plots of Figs. 1, for a fixed  $N(\text{Rb})$ , the  $^{164}\text{Dy}$  droplet can be bound for a maximum of  $N(\text{Dy})$ . This maximum of  $N(\text{Dy})$  increases with  $N(\text{Rb})$ , all other parameters remaining fixed. For a fixed  $N(\text{Rb})$ , the disk-shaped trap can accommodate a larger maximum number of  $^{164}\text{Dy}$  atoms than the cigar-shaped trap. For  $N(\text{Dy})$  smaller than this maximum number, the  $^{164}\text{Dy}$  droplet can be bound for  $|a_{12}|$  between two limiting values. For  $|a_{12}|$  above the upper limit, there is too much inter-species attraction on the  $^{164}\text{Dy}$  droplet leading to its collapse. For  $|a_{12}|$  below the lower limit, there is not enough attraction and the  $^{164}\text{Dy}$  droplet cannot be bound. The lower limit of  $|a_{12}|$  is small and tends to zero as  $N(\text{Rb})$  tends to infinity. For fixed values of  $N(\text{Rb})$  and  $N(\text{Dy})$  and for small  $|a_{12}|$  near the lower limit, the disk-shaped configuration is favored and it can bind the  $^{164}\text{Dy}$  droplet for smaller values of  $|a_{12}|$ . However, for a fixed  $N(\text{Rb})$  and for a large  $|a_{12}|$  near the upper limit, the disk-shaped configuration is favored only for a large  $N(\text{Dy})$  allowing a  $^{164}\text{Dy}$  droplet for larger  $|a_{12}|$ , whereas the cigar-shaped configuration is favored for a small  $N(\text{Dy})$  allowing a  $^{164}\text{Dy}$  droplet for larger  $|a_{12}|$ .

In Figs. 2 we illustrate the numerical and variational results for chemical potentials and root-mean-square (rms) sizes  $\langle x \rangle$  and  $\langle z \rangle$  of the trapped  $^{87}\text{Rb}$  BEC of 10000 atoms and of the bound  $^{164}\text{Dy}$  droplet versus

TABLE I: Root-mean-square sizes of the binary  $^{87}\text{Rb}$ - $^{164}\text{Dy}$  system of 50000  $^{87}\text{Rb}$  atoms from variational approximation (11) – (14) (var), and an approximate energy minimization (approx).

	$a_{12}$	$N(\text{Dy})$	$\lambda$	$\langle x_1 \rangle$	$\langle z_1 \rangle$	$\langle x_2 \rangle$	$\langle z_2 \rangle$
approx	$-40a_0$	1000	4	2.7597	0.7106	1.0755	0.4847
var	$-40a_0$	1000	4	2.7475	0.7091	1.0703	0.4834
approx	$-80a_0$	1000	0.25	2.2780	8.9525	0.6468	2.5799
var	$-80a_0$	1000	0.25	2.2549	8.8558	0.6378	2.5525
approx	$-60a_0$	500	4	2.7597	0.7106	0.8478	0.3768
var	$-60a_0$	500	4	2.7492	0.7090	0.8443	0.3757
approx	$-100a_0$	100	0.25	2.2780	8.9525	0.6384	1.5803
var	$-100a_0$	100	0.25	2.2752	8.9392	0.6373	1.5779

$N(\text{Dy})$  for (a)  $\lambda = 4$  and (b) 0.25. Considering that the  $^{164}\text{Dy}$  droplet may not have a Gaussian shape as assumed in the variational approximation, the agreement between the numerical and variational results is quite satisfactory.

The existence of the quasi-free  $^{164}\text{Dy}$  droplet can be studied qualitatively by a minimization of the energy (19). However, the widths of the  $^{87}\text{Rb}$  BEC for a fixed  $N(\text{Rb})$  do not vary much as  $N(\text{Dy})$  or  $a_{12}$  is varied. Hence for a qualitative understanding of the existence of a stable  $^{164}\text{Dy}$  droplet for a fixed  $N(\text{Rb})$ , we can make further approximation in the energy (19) and take the widths of the trapped  $^{87}\text{Rb}$  BEC to be the same as the widths of the  $^{87}\text{Rb}$  BEC in the absence of  $^{164}\text{Dy}$  atoms under otherwise identical conditions. The widths of the  $^{164}\text{Dy}$  droplet so obtained for  $N(\text{Rb}) = 50000$  are compared in Table I for several values of  $a_{12}$ ,  $N(\text{Dy})$ , and  $\lambda$  with the widths obtained from exact energy minimization. We find from Table I that the sizes of the  $^{87}\text{Rb}$  for a fixed  $N(\text{Rb})$  remains reasonably constant with respect to the variation of  $N(\text{Dy})$  and  $a_{12}$  and that the approximate energy minimization with fixed widths for  $^{87}\text{Rb}$  BEC leads to a good approximation to the widths of the  $^{164}\text{Dy}$  droplet. In the numerical solution of the binary GP equations (6) and (7) we also verified that the shape and size of the trapped  $^{87}\text{Rb}$  BEC are fairly independent of the presence or absence of the  $^{164}\text{Dy}$  droplet. Hence in the study of the shape, size and dynamics of the dipolar  $^{164}\text{Dy}$  droplet bound in the trapped  $^{87}\text{Rb}$  BEC, the trapped BEC will only have a passive role and we shall highlight only the shape, size and dynamics of the  $^{164}\text{Dy}$  droplet in the following.

We consider the shape of a stable  $^{164}\text{Dy}$  droplet in the stability plots of Figs. 1 and study its change as the collapse instability is approached. In Fig. 3 (a) we show the isodensity contour plot (density  $|\phi_1(\mathbf{r})|^2$ ) of a disk-shaped ( $\lambda = 4$ )  $^{87}\text{Rb}$  BEC of 10000 atoms and that (density  $|\phi_2(\mathbf{r})|^2$ ) of the bound  $^{164}\text{Dy}$  droplet in Figs. 3 (b) for  $N(\text{Dy}) = 1000$  ( $a_{12} = -30a_0$ ), (c) for  $N(\text{Dy}) = 1000$  ( $a_{12} = -80a_0$ ), and in (d) for  $N(\text{Dy}) = 4000$  ( $a_{12} = -30a_0$ ). The isodensity of the  $^{87}\text{Rb}$  BEC is practically the same in all three cases. Of the isodensities of



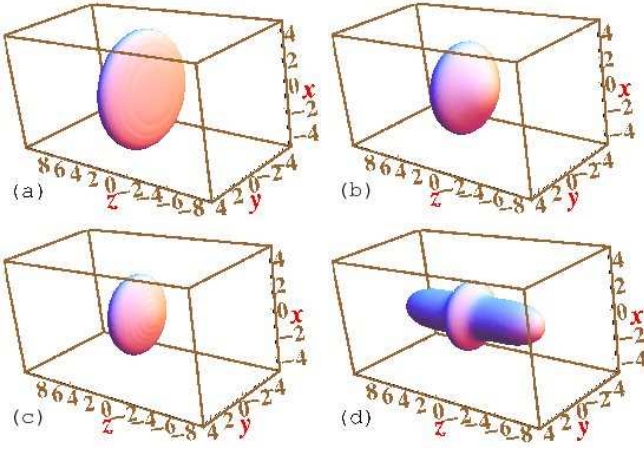


FIG. 3: (Color online) (a) The isodensity contour of a disk-shaped ( $\lambda = 4$ )  $^{87}\text{Rb}$  BEC of 10000 atoms and that of the bound  $^{164}\text{Dy}$  droplet of (b) 1000 atoms ( $a_{12} = -30a_0$ ), (c) 1000 atoms ( $a_{12} = -80a_0$ ), (d) 4000 atoms ( $a_{12} = -30a_0$ ). The isodensity contour of the  $^{87}\text{Rb}$  BEC is practically the same in all three cases. The density on the contour in all cases is  $0.001l_0^{-3}$  and length is measured in units of  $l_0$  ( $= 1 \mu\text{m}$ ).

the  $^{164}\text{Dy}$  droplet, the one in Fig. 3 (b) is deep inside the stability region far away from collapse instability. The parameters of Figs. 3 (c) and (d) are close to the region of collapse instability. Of these two, Fig. 3 (c) corresponds to a medium number of  $^{164}\text{Dy}$  atoms and Fig. 3 (d) to a large number. Independent of the initial shape, a  $^{164}\text{Dy}$  droplet with a medium number of  $^{164}\text{Dy}$  atoms always collapses towards the center with shrinking size. However, a  $^{164}\text{Dy}$  droplet containing a large number of atoms, independent of the associated trap symmetry, always first takes a cigar shape and then collapses on the axial  $z$  direction. A strong dipolar interaction prohibits a collapse to center of a large  $^{164}\text{Dy}$  droplet and favors a cigar shape. In Figs. 3 the trap is disk-shaped. The medium-sized  $^{164}\text{Dy}$  droplet of Fig. 3 (c) collapses to center maintaining the disk shape as the net dipolar interaction is smaller in this case. The large  $^{164}\text{Dy}$  droplet in Fig. 3 (d) has changed its shape from the supporting disk-shaped trap to a cigar shape. If the attraction is further increased by increasing  $|a_{12}|$ , the  $^{164}\text{Dy}$  droplet of Fig. 3 (d) would collapse from the cigar shape on the axial  $z$  axis.

In Fig. 4 (a) we illustrate the isodensity contour of a cigar-shaped ( $\lambda = 0.25$ )  $^{87}\text{Rb}$  BEC of 10000 atoms and the same of the  $^{164}\text{Dy}$  droplet are shown in Figs. 4 (b) for  $N(\text{Dy}) = 1000$  ( $a_{12} = -30a_0$ ), (c) for  $N(\text{Dy}) = 1000$  ( $a_{12} = -65a_0$ ), and in (d) for  $N(\text{Dy}) = 3000$  ( $a_{12} = -30a_0$ ). The density of the  $^{87}\text{Rb}$  BEC is virtually the same in all three cases. Of the three  $^{164}\text{Dy}$  droplets, the one in Fig. 4 (b) is deep inside the stability region far away from collapse instability. The parameters for Figs. 4 (c) and (d) are close to the region of collapse instability.

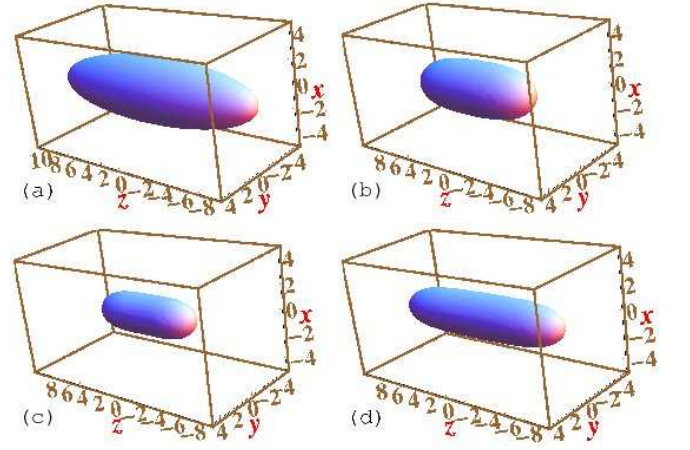


FIG. 4: (Color online) (a) The isodensity contour of a cigar-shaped ( $\lambda = 0.25$ )  $^{87}\text{Rb}$  BEC of 10000 atoms and that of the bound  $^{164}\text{Dy}$  droplet of (b) 1000 atoms ( $a_{12} = -30a_0$ ), (c) 1000 atoms ( $a_{12} = -65a_0$ ), (d) 3000 atoms ( $a_{12} = -30a_0$ ). The isodensity contour of the  $^{87}\text{Rb}$  BEC is practically the same in all three cases. The density on the contour in all cases is  $0.001l_0^{-3}$  and length is measured in units of  $l_0$  ( $= 1 \mu\text{m}$ ).

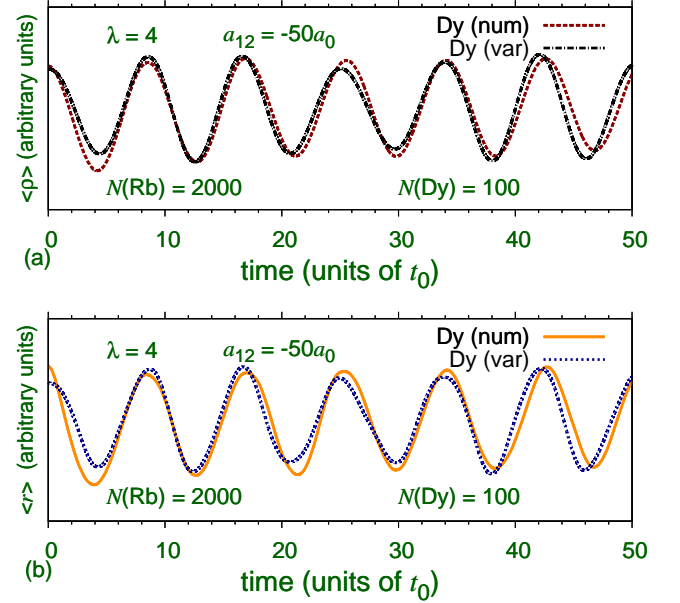


FIG. 5: (Color online) The rms sizes (a)  $\langle \rho \rangle$  and (b)  $\langle r \rangle$  from numerical simulation (num) and variational approximation (var) during breathing oscillation initiated by the sudden change  $a_{12} \rightarrow 1.01 \times a_{12}$  versus time in units of  $t_0$  ( $= 1.38$  ms) of a  $^{164}\text{Dy}$  droplet of 100 atoms bound in a trapped  $^{87}\text{Rb}$  BEC of 2000 atoms for  $a_{12} = -50$  and  $\lambda = 4$ .

In all cases the  $^{164}\text{Dy}$  droplet maintains the cigar shape of the accompanying trap acting on the  $^{87}\text{Rb}$  BEC. The size of the  $^{164}\text{Dy}$  droplet in Fig. 4 (b) is larger than that

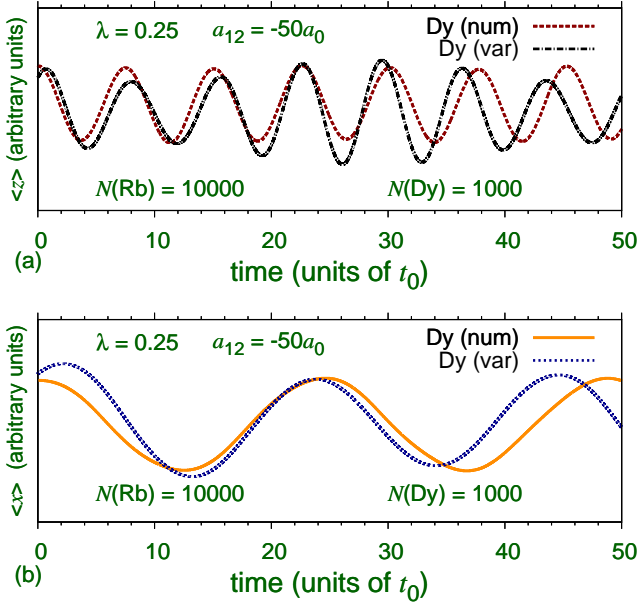


FIG. 6: (Color online) The rms sizes (a)  $\langle z \rangle$  and (b)  $\langle x \rangle$  from numerical simulation (num) and variational approximation (var) during breathing oscillation initiated by the sudden change  $a_{12} \rightarrow 1.01 \times a_{12}$  versus time in units of  $t_0$  ( $= 1.38$  ms) of a  $^{164}\text{Dy}$  droplet of 1000 atoms bound in a trapped  $^{87}\text{Rb}$  BEC of 10000 atoms for  $a_{12} = -50$  and  $\lambda = 0.25$ .

of Fig. 4 (c) with the same number of atoms due to a strong inter-species attraction acting on the latter. The  $^{164}\text{Dy}$  droplet of Fig. 4 (d) is larger than those in Figs. 4 (b) and (c) due to a larger number of  $^{164}\text{Dy}$  atoms in it.

We investigate the dynamics of a bound  $^{164}\text{Dy}$  droplet in a trapped  $^{87}\text{Rb}$  BEC. First, to test the present scheme we consider a small  $^{164}\text{Dy}$  droplet of 100 atoms bound in a disk-shaped ( $\lambda = 4$ )  $^{87}\text{Rb}$  BEC of 2000 atoms with  $a_{12} = -50a_0$ . This corresponds to the stable region of Fig. 1 (a). In real-time evolution of the system, the scattering length  $a_{12}$  is suddenly changed to  $1.01 \times a_{12}$  thus starting the breathing oscillation. In Figs. 5 we show the resultant oscillation of the rms sizes (a)  $\langle \rho \rangle$  and (b)  $\langle r \rangle$  during time evolution as obtained from numerical solution of Eqs. (6) and (7) and variational approximation (11) – (14).

After obtaining a satisfactory result of dynamics with a small system, we venture with a larger system of experimental interest, e. g., a stable  $^{164}\text{Dy}$  droplet of 1000 atoms bound in a cigar-shaped ( $\lambda = 0.25$ )  $^{87}\text{Rb}$  BEC of 10000 atoms with  $a_{12} = -50a_0$  corresponding to the stable region of Fig. 1 (b). In real-time evolution of the system, the scattering length  $a_{12}$  is again suddenly changed to  $1.01 \times a_{12}$  thus starting the breathing oscillation. In Figs. 6 we show the resultant oscillation of the rms sizes (a)  $\langle z \rangle$  and (b)  $\langle x \rangle$  during time evolution as obtained from a numerical solution of Eqs. (6) and (7)

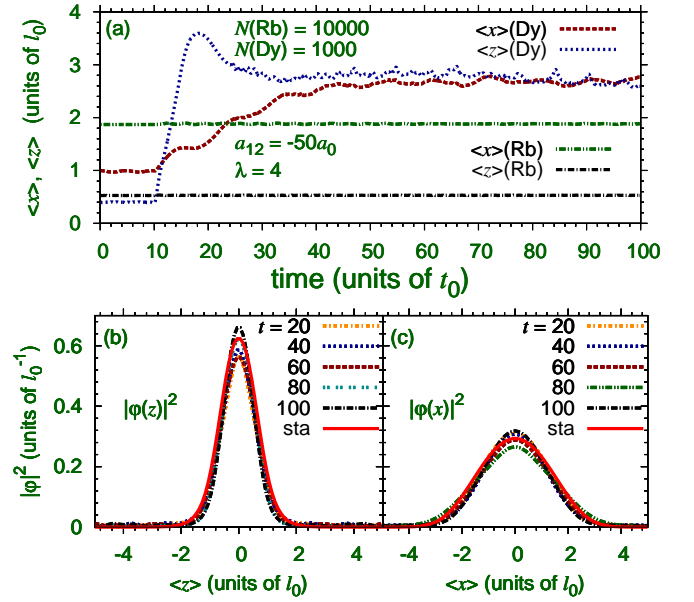


FIG. 7: (Color online) (a) The rms sizes  $\langle x \rangle$  and  $\langle z \rangle$  of the  $^{164}\text{Dy}$  and  $^{87}\text{Rb}$  BECs versus time during the passage of the trapped  $^{164}\text{Dy}$  BEC in the binary  $^{87}\text{Rb}$ - $^{164}\text{Dy}$  mixture with  $N(\text{Rb}) = 10000$ ,  $N(\text{Dy}) = 1000$ ,  $a_{12} = -50a_0$  and  $\lambda = 4$  as the confining trap on the  $^{164}\text{Dy}$  BEC is relaxed for  $t > 10$  exponentially as  $V_{\text{trap}} \rightarrow \exp[-(t-10)]V_{\text{trap}}$ , so that this trap is practically zero for  $t > 20$ . The linear 1D densities (b)  $|\varphi(z)|^2$  and (c)  $|\varphi(x)|^2$  at times  $t = 20, 40, 60, 80, 100$  (in units of  $t_0$ ) during the expansion together with the numerically calculated density of the stationary (sta)  $^{164}\text{Dy}$  droplet.

and variational approximation (11) – (14). The agreement between the numerical simulation and variational approximation in both cases of breathing oscillation illustrated in Figs. 5 and 6 for disk- and cigar-shapes is quite satisfactory considering the facts that (i) during oscillation the bound  $^{164}\text{Dy}$  droplet might have a density distribution different from a Gaussian distribution assumed in the variational approximation and (ii) the large non-linearity in the  $^{87}\text{Rb}$  BEC will also make its density distribution non-Gaussian. The stable oscillation under small perturbation as shown in Figs. 5 and 6 confirms the dynamical stability of the  $^{164}\text{Dy}$  droplet for both disk and cigar type environments.

The present quasi-free dipolar droplet is not just of theoretical interest, but can be realized experimentally by initially preparing a binary  $^{87}\text{Rb}$ - $^{164}\text{Dy}$  mixture where both the components are harmonically trapped. The trap on  $^{164}\text{Dy}$  can then be ramped to zero exponentially during a few milliseconds. The  $^{164}\text{Dy}$  cloud will initially expand and finally form the quasi-free  $^{164}\text{Dy}$  droplet. To illustrate numerically the viability of this procedure, we consider an initial  $^{87}\text{Rb}$ - $^{164}\text{Dy}$  binary mixture with following parameters:  $N(\text{Rb}) = 10000$ ,  $N(\text{Dy}) = 1000$ ,  $a_{12} = -50a_0$ . We take the same axial trap with  $\lambda = 4$  acting on both components. Then we per-

form real-time propagation of the binary GP equation with this initial state and ramp down the  $^{164}\text{Dy}$  trap by  $V_{\text{trap}}(\text{Dy}) \rightarrow \exp[-(t-10)]V_{\text{trap}}(\text{Dy})$  for time  $t > 10$ . The trap on  $^{164}\text{Dy}$  is practically zero for times  $t \geq 20$ . The trapped  $^{164}\text{Dy}$  BEC first expands for  $t > 10$  and eventually it emerges as the quasi-free  $^{164}\text{Dy}$  droplet. The time evolution of the rms sizes of the  $^{164}\text{Dy}$  droplet is shown in Fig. 7 (a). The widths of the  $^{164}\text{Dy}$  droplet first increase and eventually execute small oscillation illustrating the stable  $^{164}\text{Dy}$  droplet. To investigate the shape of the  $^{164}\text{Dy}$  droplet, we plot in Figs. 7 (b) and (c) the one-dimensional (1D) densities  $|\varphi(z)|^2 \equiv \int \int dx dy |\phi(x, y, z)|^2$  and  $|\varphi(x)|^2 \equiv \int \int dy dz |\phi(x, y, z)|^2$  at times  $t = 20, 40, 60, 80$ , and 100 along with the corresponding numerical densities of the stationary droplet calculated from the solution of Eqs. (6) and (7). The 1D numerical densities of the stationary  $^{164}\text{Dy}$  droplet are in agreement with the densities obtained from dynamical simulation of the passage of the trapped  $^{164}\text{Dy}$  BEC to a quasi-free droplet. In Figs. 7 (b) and (c) one can easily identify the small oscillation of the evolving dynamical droplet around its stationary shape.

#### IV. SUMMARY AND DISCUSSION

Using variational approximation and numerical solution of a set of coupled mean-field GP equations, we demonstrate the existence of a stable dipolar  $^{164}\text{Dy}$  droplet bound by inter-species attraction in a trapped non-dipolar BEC of  $^{87}\text{Rb}$  atoms. The domain of stability of the  $^{164}\text{Dy}$  droplet is highlighted in stability plots of number of  $^{164}\text{Dy}$  atoms and inter-species scattering length  $a_{12}$  for both cigar- and disk-shaped traps acting on the  $^{87}\text{Rb}$  BEC. Results of variational approximation and numerical solution for the statics (sizes and chemical potentials) and dynamics (breathing oscillation) of the  $^{164}\text{Dy}$  droplet are found to be in satisfactory agreement with each other. We also demonstrate numerically that such droplets can be obtained experimentally by considering a trapped binary  $^{87}\text{Rb}$ - $^{164}\text{Dy}$  BEC and then removing the trap on  $^{164}\text{Dy}$ . The  $^{164}\text{Dy}$  BEC then expands and transforms into a bound dipolar droplet.

Dipolar interaction among atoms is quite different from normal short-range atomic interaction and manifests in different ways in a trapped dipolar BEC. However, in a trapped dipolar BEC, the confining constraints could be too strong and could make the effect of dipolar inter-

action difficult to observe. Special experimental set up and theoretical formulation might be necessary to study the effect of dipolar interaction. However, it will be much easier to see the effect of dipolar interaction in the present binary mixture of non-dipolar  $^{87}\text{Rb}$  and dipolar  $^{164}\text{Dy}$ . Strong dipolar interaction in the axial polarization  $z$  direction should elongate the dipolar BEC along this direction thus transforming it to a cigar shape. However, it will be difficult to observe this change in the presence of a harmonic trap along  $z$ . The effect of dipolar interaction is clearly seen in the present quasi-free  $^{164}\text{Dy}$  droplet of Fig. 3 (d) where due to a strong dipolar interaction the shape of the  $^{164}\text{Dy}$  droplet has changed from the disk shape – the shape of the  $^{87}\text{Rb}$  BEC of Fig. 3 (a) responsible for its binding – to the cigar shape. The thin disk-shaped  $^{87}\text{Rb}$  BEC stays only near the central  $z = 0$  plane of the  $^{164}\text{Dy}$  droplet. Most of the  $^{164}\text{Dy}$  atoms lying outside the  $^{87}\text{Rb}$  BEC are bound due to the intra-species dipolar interaction. In the dipolar  $^{164}\text{Dy}$  droplet of Fig. 3 (d) the dipolar interaction is playing a more important role in its binding and shape.

The present quasi-free dipolar droplet may also find other interesting applications in the BEC phenomenology. Among the interesting features in a trapped dipolar BEC, one can mention its peculiar shape and stability properties [11], and D-wave collapse [19], anisotropic soliton, vortex soliton [20] and vortex lattice [21], anisotropic shock and sound wave propagation [22], anisotropic Landau critical velocity [23], stable checkerboard, stripe, and star configurations in a two-dimensional (2D) optical lattice as stable Mott insulator [24] as well as superfluid soliton [25] states. It would be of great interest to find out how these features and properties of a trapped dipolar BEC would manifest in a 3D quasi-free dipolar droplet. For example, a quasi-free dipolar droplet could be used in the experimental study of anisotropic sound and shock-wave propagation [22], collapse dynamics [19], anisotropic Landau critical velocity [23], formation of vortex dipoles and vortex lattice [21] etc in a different setting of confinement which will facilitate the observation of the effect of anisotropic dipolar interaction.

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